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The Potts model on Bethe lattices: I. General results

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Abstract. The q-state ferromagnetic Potts model (FPM) and antiferromagnetic Potts model (APM) are solved on Bethe lattices for all values of the external magnetic field and temperature. The exact expressions of all thermodynamic functions of interest in the FPM and APM are calculated. We find the complete phase diagrams for both systems. In the FPM there are first-order phase transitions at the critical point for every $q > 2$. In the APM we find second-order phase transitions along a critical line for every $q \geq 2$.

1. Introduction

The Potts model (Potts 1952) has received an increasing theoretical and experimental interest in recent years, and at present a great many rigorous and approximate results are known. An extended summary of results and bibliography can be found in the review article by Wu (1982) and references therein. The model can be studied on Bethe lattices, i.e. infinite connected trees whose sites have the same coordination number $\sigma+1$ (see figure 1). Compared with standard lattices, the Bethe lattices are

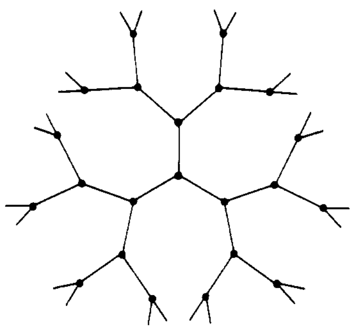

Figure 1. Bethe lattice of coordination number $\sigma+1=3$.

topological abstractions: however, under certain conditions, exact results on them correspond to approximated results on standard lattices. Here, the Bethe lattices are fruitfully used to have a deeper insight into the behaviour of the ferromagnetic Potts model (FPM) and antiferromagnetic Potts model (APM).

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All methods of solution of Hamiltonian models on Bethe lattices implicitly make use of probability measures: in terms of them we can distinguish two distinct approaches. In the conventional thermodynamic limit, surface effects are not negligible, and one finds a probability measure which privileges a 'central' site of the lattice and distinguishes sites belonging to the various shells surrounding it. In this case the system exhibits a special type of phase transitions (see Eggarter (1974), Müller-Hartmann and Zittartz (1974, 1975) for the ferromagnetic and antiferromagnetic Ising model; Wang and Wu (1976) for the FPM; Baumgärtel and Müller-Hartmann (1982) for the random cluster model) which however is not seen in the behaviour of standard lattices. If one wants to reproduce their features, one has to find a probability measure which is translationally invariant on the Bethe lattice (such are, as a matter of fact, the typical probability measures on real lattices). This last choice has also been made by Wang and Wu (1976), who found a divergence of the susceptibility at a temperature they show to be the Bethe–Peierls temperature T_{BP} on real lattices.

In this paper we follow the second procedure, and show that T_{BP} individuates one of the two spinodal points that the FPM exhibits at zero external field: the system undergoes a first-order phase transition at an intermediate critical temperature. Moreover, we treat the FPM and APM at non-zero external field, and derive their properties in full detail.

The method we introduce for the solution is based essentially on (a) the use of tools of measure theory, and (b) the determination of the free energy. As regards (a), it can be proved that a Markovian probability measure, which is rotationally and translationally invariant on a Bethe lattice, is completely defined by a small number of fundamental probabilities. Then, (b) these probabilities are explicitly determined and used to compute the entropy and the internal energy of the system, which give the free energy. The mathematical details of (a) and (b) are introduced by Peruggi (1983): here we focus mainly on the physical results. Using the analytical expression of the free energy, we obtain the thermodynamic functions of interest and study their behaviour. In this paper we give a general review of the FPM and APM properties for $q \geqslant 2$. Special topics, such as the specific and latent heats, the asymptotic behaviour ($q \to \infty$, $q \to 1$, $q \to 0$), the case $q < 2$, and critical exponents, will be discussed in a subsequent paper.

The present method can be easily applied to other related problems such as standard or AB Potts-correlated-site/random-bond percolation problems on Bethe lattices (Peruggi 1983, Peruggi *et al* 1983), and the FPM with quenched site-dilution, whose study is actually in progress.

The outline of the paper is as follows. In § 2 we introduce the general terminology and calculate the most important thermal functions of the FPM, whose properties are studied and discussed in § 3. The APM is introduced in § 4, and the proper functions are evaluated; their behaviour is deduced and described in § 5. Finally, in § 6 we summarise and criticise the results.

2. The ferromagnetic model thermal functions

Let us consider a Bethe lattice L, whose sets of sites and bonds will be denoted as V and E, respectively (for graph theory terminology we refer to Essam and Fisher (1970)). We introduce a regular sequence $\{T_n\}_{n=1}^{\infty}$ of finite trees such that $T_n \subset T_{n+1}$ for every index n, and $\bigcup_{n=1}^{\infty} T_n = L$. We associate a q-state variable ν_i to each site

$i \in V$, and for every tree $T_n = (V_n, E_n)$ let all variables interact according to a ferromagnetic ($K > 0$) Potts Hamiltonian:

$$-\beta\mathscr{H}_n \equiv K \sum_{\langle ij\rangle \in E_n} \delta_{\nu_i\nu_j} + H \sum_{i \in V_n} \delta_{\nu_i 1}. \tag{1}$$

Starting from the probability measure μ_n, induced by the Hamiltonian (1) on T_n, we can use the standard limit procedures to determine a probability measure μ on L. In analogy with real lattices we require that

(i) μ is rotationally and one-step translationally invariant on L.

Following Coniglio (1976), property (i) will be obtained by imposing that

$$\lim_{n\to\infty} \mu_n(\nu_u = 1) = \lim_{n\to\infty} \mu_v(\nu_n = 1) \tag{2}$$

where u is the common central site of the trees T_n, and v is one of its adjacent sites. We also consider translational invariance of the lattice itself, i.e. when taking the thermodynamic limits we set

$$\lim_{n\to\infty} (|E_n|/|V_n|) = \tfrac{1}{2}(\sigma + 1) \tag{3}$$

where $|E_n|$ and $|V_n|$ are, respectively, the number of bonds and sites of T_n.

Property (i) of μ, together with the property

(ii) μ is one-step Markov on L,

induced by the Hamiltonian (1), allow us to show that the probability measure of any event on L is completely determined by the knowledge of elementary probabilities which we denote p_r and p_{rs}, with $r, s = 1, \ldots, q$. Here $p_r \equiv \mu(\nu_i = r)$ and $p_{rs} \equiv \mu(\nu_j = s|\nu_i = r) \equiv$ conditional probability that $\nu_j = s$, provided that $\nu_i = r$, for every $i \in V$ and $j \in V$ such that $\langle ij\rangle \in E$. Using the equivalence of the spin states $r = 2, \ldots, q$ and the normalisation conditions

$$\sum_{r=1}^{q} p_r = 1, \qquad \sum_{s=1}^{q} p_{rs} = 1, \qquad r = 1, \ldots, q, \tag{4}$$

it is easy to see that only four elementary probabilities are independent. For our purposes we chose

$$\begin{aligned} &p_1 = 1/[1 + (q-1)\varphi\bar{\varphi}(\varphi)], \qquad p_{11} = e^K/a(\varphi), \\ &p_{21} = 1/b(\varphi), \qquad p_{22} = e^K\varphi/b(\varphi), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &a(\varphi) \equiv e^K + (q-1)\varphi, \qquad b(\varphi) \equiv 1 + (e^K + q - 2)\varphi, \\ &\bar{\varphi}(\varphi) \equiv b(\varphi)/a(\varphi). \end{aligned} \tag{6}$$

Finally, for any choice of H and K the parameter φ can be computed by means of relation (2), which in terms of φ is

$$\varphi = e^{-H}\bar{\varphi}^{\sigma}(\varphi). \tag{7}$$

The free energy of the system is obtained using its definition:

$$\beta\mathscr{F} \equiv \beta\mathscr{U} - \mathscr{S}/k \tag{8}$$

where k is the Boltzmann constant. Note that, here and in the following, the extensive functions are calculated per site of the lattice. The internal energy is obviously given by

$$\beta \mathcal{U} = -Hp_1 - \tfrac{1}{2}(\sigma+1)K \sum_{r=1}^{q} p_r p_{rr} \tag{9}$$

while the entropy can be obtained by direct counting of the possible realisations of the system:

$$\mathcal{S} = k\left(\sigma \sum_{r=1}^{q} p_r \ln p_r - \frac{\sigma+1}{2} \sum_{r,s=1}^{q} p_r p_{rs} \ln(p_r p_{rs})\right). \tag{10}$$

Observe that relation (10) is the same as the entropy evaluated by Kikuchi (1951, 1970) in the first-order approximation of the cluster variation method. Finally we have the magnetisation and the susceptibility of the system:

$$\mathcal{M} \equiv -\partial(\beta\mathcal{F})/\partial H = p_1, \tag{11a}$$

$$\chi \equiv \frac{\partial^2}{\partial H^2}(\beta\mathcal{F}) = p_1(1-p_1)\frac{1+(p_{11}-p_{21})}{1-\sigma(p_{11}-p_{21})}, \tag{11b}$$

where equation (7) has been used to evaluate $\partial\varphi/\partial H$.

Now, let i and j be two sites of L at a distance l ($\equiv l(i,j) \equiv$ number of bonds connecting i to j) which are respectively in the states r and s. Then the correlation function is given by

$$\begin{aligned}\mathcal{G}_{ij}(r,s) &\equiv \langle\delta_{\nu_i r}\delta_{\nu_j s}\rangle - \langle\delta_{\nu_i r}\rangle\langle\delta_{\nu_j s}\rangle \\ &= p_r \sum_{\nu_x=1}^{q}\sum_{\nu_y=1}^{q}\cdots\sum_{\nu_z=1}^{q} p_{r\nu_x}p_{\nu_x\nu_y}\cdots p_{\nu_z s} - p_r p_s \\ &= p_r(\langle r|\mathbf{T}^l|s\rangle - p_s) \\ &= \mathcal{G}_l(r,s)\end{aligned} \tag{12}$$

where $x, y, \ldots, z$ are the sites of the unique path on L between i and j, and the $q\times q$ transfer matrix $\mathbf{T}$ is defined as

$$\langle r|\mathbf{T}|s\rangle \equiv p_{rs}. \tag{13}$$

The evaluation of the elements of $\mathbf{T}^l$ can be reduced to the diagonalisation of a 2×2 matrix, and gives (Peruggi 1983)

$$\mathcal{G}_l(r,s) = \frac{1-p_1}{q-1}\left[\varepsilon(r,s)p_1(p_{11}-p_{21})^l + \bar{\varepsilon}(r,s)\left(\frac{(q-1)p_{22}+p_{21}-1}{q-2}\right)^l\right] \tag{14}$$

where

$$\begin{aligned}\varepsilon(r,s) &= q^2\delta_{r1}\delta_{1s} - q(\delta_{r1}+\delta_{1s}) + 1, \\ \bar{\varepsilon}(r,s) &= (q-1)\delta_{rs} - q\delta_{r1}\delta_{1s} + (\delta_{r1}+\delta_{1s}) - 1.\end{aligned} \tag{15}$$

Remark that by means of the fluctuation relation

$$\chi = \sum_{j\in V} \mathcal{G}_{ij}(1,1) \tag{16}$$

through equation (14) we recover (11b).

3. The ferromagnetic model critical properties

In this section we study the behaviour of the system by solving relation (7). It can be shown that this equation admits at least one solution, and that two other solutions appear for certain ranges of H and K. To give the most direct physical interpretation we do not consider the values of φ, but the corresponding magnetisation. We also use the free energy, for fixed H and K, written as a function of the possible values of the magnetisation. This can easily be done using relations (4), (5), (6) to eliminate φ and to write all elementary probabilities as functions of p_1, and finally inserting these expressions in (8). The solutions of (7), for given H and K, individuate values of p_1 which are in one-to-one correspondence with maxima, minima, and flexes of $\mathscr{F}(p_1; H, K)$. In the magnetisation plots of figures 2 and 3 the following code is used. Full lines represent stable states corresponding to the deeper relative ($\equiv$absolute) minimum/a of $\mathscr{F}$. Broken lines represent metastable states corresponding to the upper relative minimum (if any) of $\mathscr{F}$. Dotted lines represent unstable states corresponding to the relative maximum (if any) of $\mathscr{F}$: in this case we have $\chi < 0$, too. Note that the points where stable or metastable states coincide with unstable states correspond to flexes of $\mathscr{F}$: these are spinodal points where the susceptibility diverges. The magnetisation is plotted as a function of the reduced temperature $\theta \equiv e^{-|K|}$, for fixed values of the reduced magnetic field $h \equiv H/|K|$ (which we introduce to eliminate the temperature dependence of H).

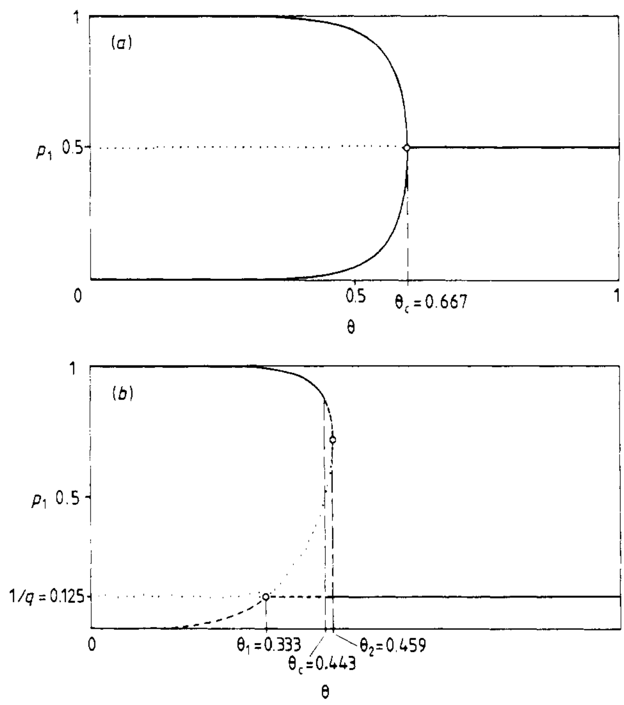

Figure 2. Magnetisation versus reduced temperature plots at $h = 0$, for $\sigma = 5$, $q = 2$ (a) and $q = 8$ (b). Stable states of the system are represented by full lines, metastable states by broken lines, and unstable states by dotted lines. Open circles indicate the points where the susceptibility diverges. The curves reported here are typical for the $q = 2$ and $q > 2$ cases (for every $\sigma > 1$).

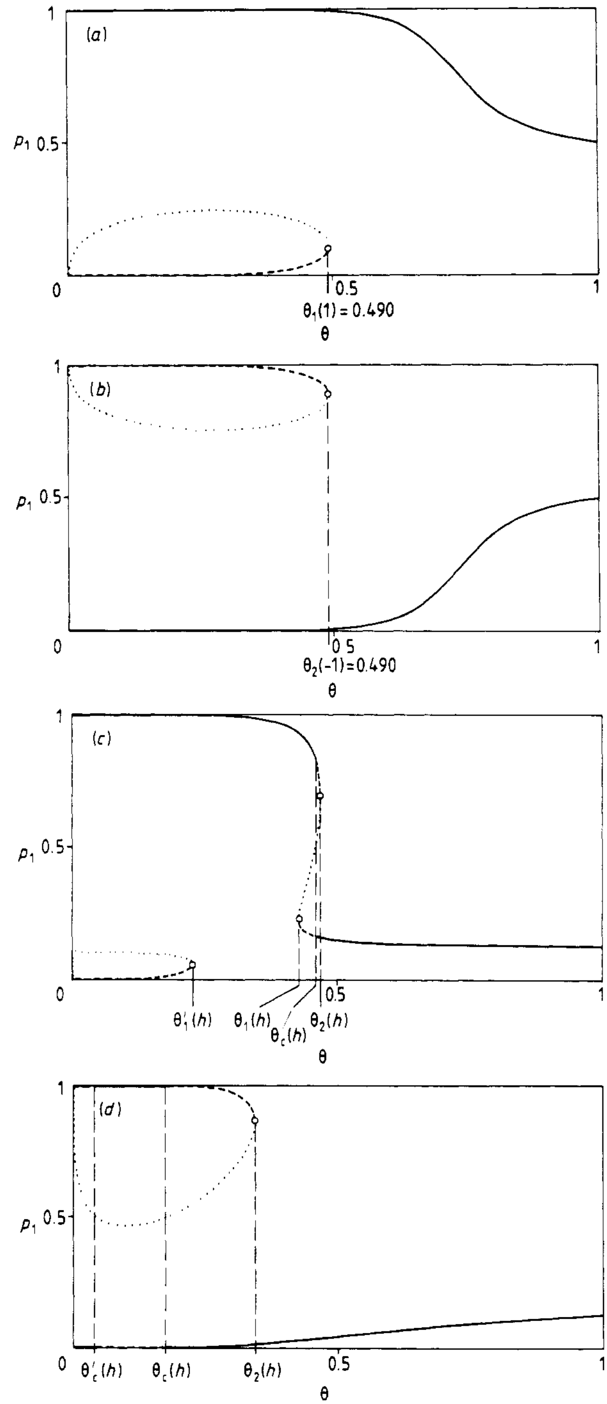

Figure 3. Magnetisation versus reduced temperature plots at constant reduced magnetic field. The Ising model, $q = 2$, is reported in (a) for $0 < h < \sigma - 1$ and in (b) for $-(\sigma - 1) < h < 0$. The Potts model, $q > 2$, is given in (c) for $0 < h < h^*$, and in (d) for $\bar{h} < h < 0$. The same code as in figure 2 is used. The actual values used for the plots are $\sigma = 5$, $q = 2$, $|h| = 1$ (a), (b); $q = 8$, $h = 0.1$ (c); $q = 8$, $h = -1$ (d).

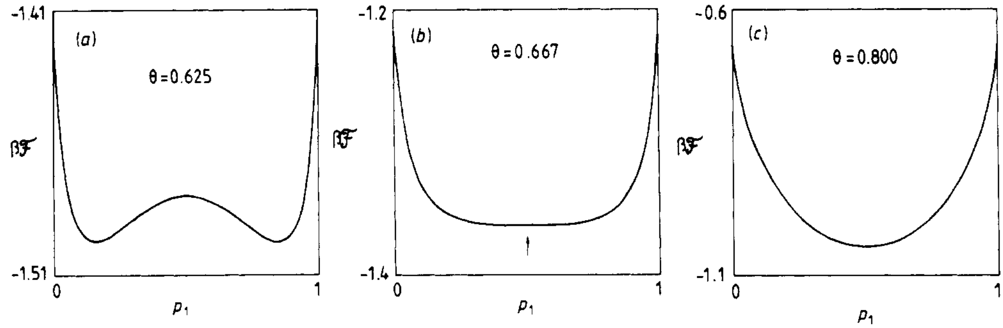

Figure 4. Free energy versus magnetisation plots in the Ising model at zero external field. The diagrams are drawn for $\sigma = 5$ and the values of θ indicated. The origin of the vertical axis is conveniently shifted and the scale expanded. An arrow indicates the position of the flex of $\beta\mathcal{F}$. These figures are chosen as representative of the cases $q = 2$; $\sigma > 1$; $\theta < \theta_c$ (a), $\theta = \theta_c$ (b), $\theta > \theta_c$ (c).

Let us consider now the single cases in detail. We start from the simplest case $h = 0$. For $q = 2$ we have the well known Ising model (see figure 2(a)) for which the free energy versus magnetisation plots are given in figure 4. Note that below the critical temperature $\theta_c = (\sigma - 1)/(\sigma + 1)$ the two minima of $\mathcal{F}$ are always equal, giving two coexisting stable states which can be reached in the limits $h \to 0^+$ and $h \to 0^-$. For $q > 2$ we have a substantially different situation (see figures 2(b)–5). Now, the two minima of $\mathcal{F}$ which appear for $\theta < \theta_2$ are not equal, so metastable states arise. We have equality only at the critical temperature

$$\theta_c = [(q-1)^{(\sigma-1)/(\sigma+1)} - 1]/(q-2) \tag{17}$$

where the system undergoes a first-order phase transition with a magnetisation jump from $p_1 = 1/q$ to $p_1 = (q-1)/q$. According to the order parameter definition for the Potts model

$$m \equiv (q\mathcal{M} - 1)/(q-1) \tag{18}$$

this corresponds to a jump from $m = 0$ to $m = (q-2)/(q-1)$. At zero external field we also find two spinodal points which represent respectively the boundary of the metastable disordered ($\equiv$supercooled) phase at $\theta_1 < \theta_c$, and the boundary of the metastable ordered ($\equiv$superheated) phase at $\theta_2 > \theta_c$.

The most interesting situations arise for sufficiently small $h \neq 0$, and low temperatures. The behaviour of the Ising model is plotted in figures 3(a) and 3(b) for $h > 0$ and $h < 0$ respectively. Note the symmetry by change of sign of h, and the spinodal points at temperatures $\theta_1(+|h|) = \theta_2(-|h|)$, which go to θ_c for $h \to 0$. For $q > 2$ the symmetry breaks up and we have the behaviour represented in figure 3(c) ($h > 0$) and 3(d) ($h < 0$). For sufficiently small $h > 0$ we find three spinodal points and one critical temperature such that $\theta_1'(h) < \theta_1(h) < \theta_c(h) < \theta_2(h)$, and $\theta_1'(0) = \theta_1(0) = \theta_1$, $\theta_c(0) = \theta_c$, $\theta_2(0) = \theta_2$. The reduced temperature $\theta_1'(h)$ is a decreasing function which goes to 0 for $h \to \sigma - 1$, while $\theta_1(h)$, $\theta_c(h)$, and $\theta_2(h)$ are increasing functions which go to the common limit θ^* for $h \to h^* > 0$. For $h < 0$ and sufficiently small $|h|$ we find one spinodal point and two critical temperatures such that $\theta_c'(h) < \theta_c(h) < \theta_2(h)$, and $\theta_c'(0) = 0$, $\theta_c(0) = \theta_c$, $\theta_2(0) = \theta_2$. The reduced temperature $\theta_2(h)$ is an increasing function which goes to 0 for $h \to -(\sigma - 1)$; while $\theta_c'(h)$ and $\theta_c(h)$ are, respectively, decreasing and increasing functions which go to the common limit $\bar{\theta}$ for $h \to \bar{h} < 0$. Finally, we

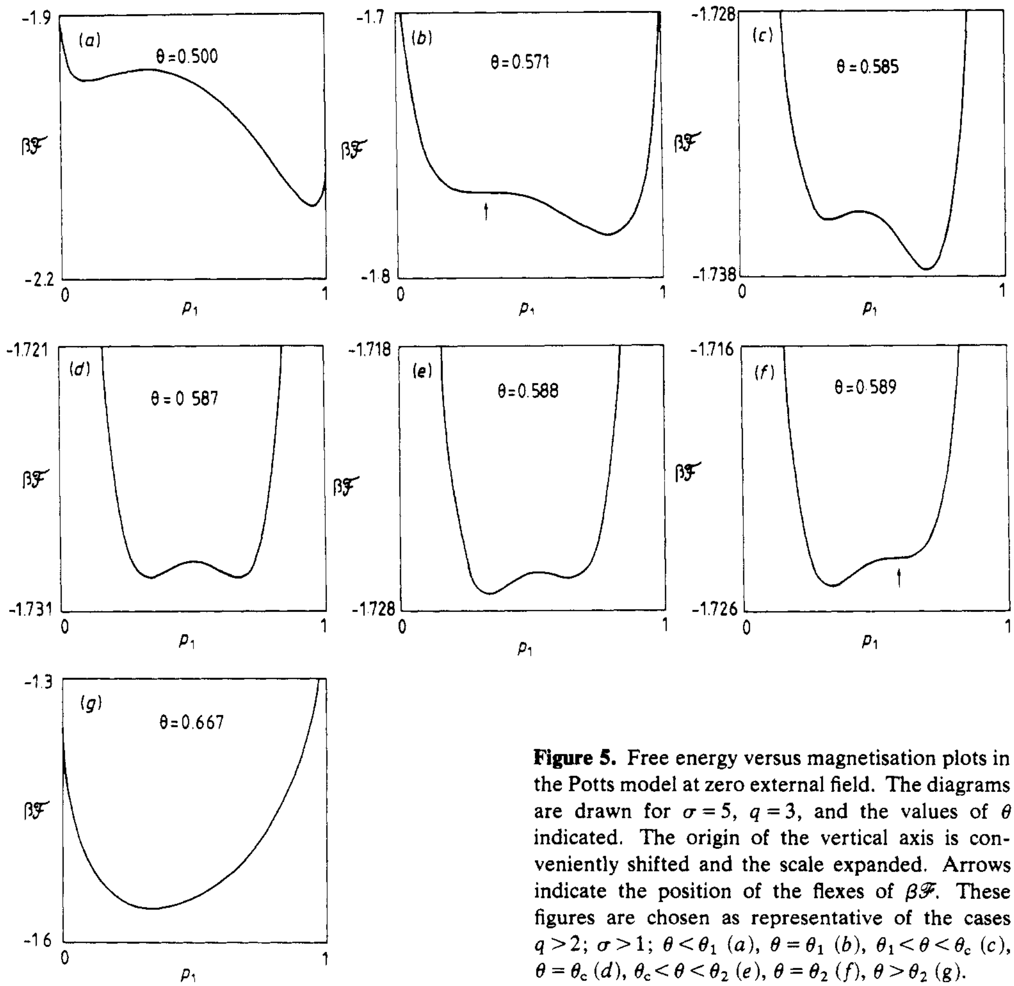

Figure 5. Free energy versus magnetisation plots in the Potts model at zero external field. The diagrams are drawn for $\sigma=5$, $q=3$, and the values of θ indicated. The origin of the vertical axis is conveniently shifted and the scale expanded. Arrows indicate the position of the flexes of $\beta\mathcal{F}$. These figures are chosen as representative of the cases $q>2$; $\sigma>1$; $\theta<\theta_1$ (*a*), $\theta=\theta_1$ (*b*), $\theta_1<\theta<\theta_c$ (*c*), $\theta=\theta_c$ (*d*), $\theta_c<\theta<\theta_2$ (*e*), $\theta=\theta_2$ (*f*), $\theta>\theta_2$ (*g*).

note that the magnetisation jump at $\theta_c'(h)$ and $\theta_c(h)$ is a decreasing function of the temperature which goes to 1 for $\theta\to 0$ and goes to 0 for $\theta\to\theta^*$. This means that we have first-order transitions all along the phase coexistence line $h_c(\theta)$ ($\equiv$ the function whose inverse is the double-valued function with branches $\theta_c'(h)$ and $\theta_c(h)$), except at θ^*, h^* where the system undergoes a second-order phase transition. For $\theta>\theta^*$ the functions $\mathcal{M}$ and χ are continuous and phase transitions disappear. The critical line is given by

$$h_c(\theta)=(2\ln\theta)^{-1}\{(\sigma-1)\ln(q-1)-(\sigma+1)\ln[1+(q-2)\theta]\} \tag{19}$$

and its terminal point is situated at the temperature

$$\theta^*=2[\![-(q-2)+\{(q-2)^2+4(q-1)[(\sigma+1)/(\sigma-1)]^2\}^{1/2}]\!]^{-1}. \tag{20}$$

Remark that the Ising model is recovered for $q\to 2$, since one has $h_c(\theta)\equiv 0$, while (17) and (20) give $\theta_c=\theta^*=(\sigma-1)/(\sigma+1)$.

All the preceding results can be fruitfully regarded as phase diagrams in the h, θ plane: see figure 6, where full lines represent phase coexistence lines, while spinodal lines are broken. Observe that the Ising curves (figure 6(*a*)) and the Potts curves

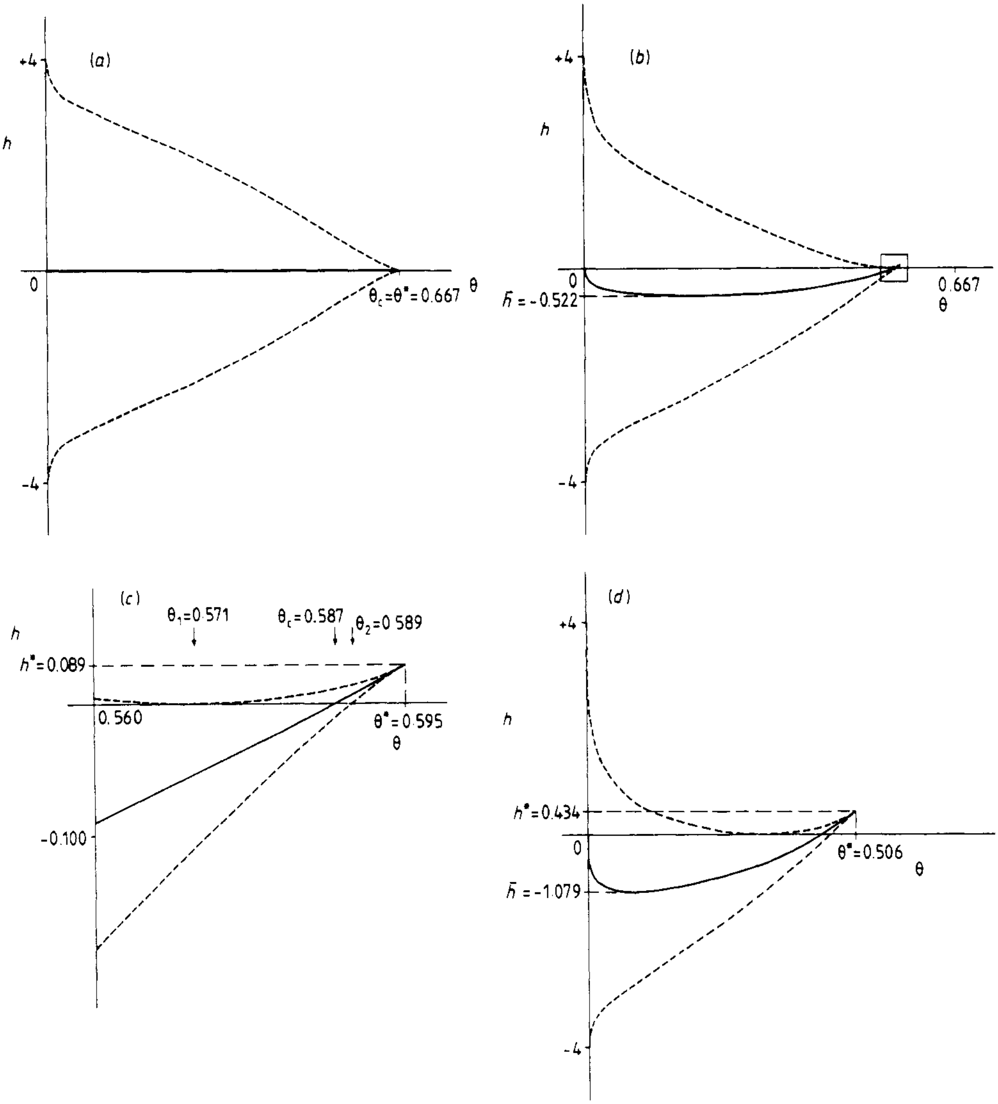

Figure 6. Phase diagrams of the FPM in the h, θ plane. The curves are drawn for $\sigma = 5$, and $q = 2$ (a), $q = 3$ (b), (c), $q = 8$ (d). Diagram (c) gives the details enclosed in the square in (b). The spinodal curves (broken lines) correspond to virtual first-order transitions (with diverging susceptibility) from a metastable state to the stable state of the system. To every point within the spinodal lines there correspond three (respectively unstable, metastable, stable) states of the system. The role of stable and metastable states interchanges crossing the phase coexistence line (full line), where they have the same free energy and the system undergoes a first-order transition. A second-order transition appears only at the point h^*, θ^* where the spinodal curves and the phase coexistence line join together. Note that h^* is an increasing function of q.

(figures 6(b,c,d)) are the same except for scale factors and a deformation due to the symmetry breaking between 'up' ($\equiv 1$) and 'down' ($\equiv$all other) spin states. One relevant feature of the Potts case ($q > 2$) is the temperature range $0 < \theta < \theta_c$ where stable states appear which have magnetisation 'opposite' to the field.

Let us now review some known results and compare them with the present solution. Wang and Wu (1976) found the Bethe–Peierls temperature T_{BP} on real lattices and demonstrated that the zero-field susceptibility on Bethe lattices diverges at T_{BP}. Southern and Thorpe (1979) solved a dilute random-bond FPM on Bethe lattices. They use the divergence of the susceptibility to find the phase boundary in the concentration–temperature plane, and obtain, at zero dilution, the same value for T_{BP}. On the other hand, mean field theory (MFT) of the FPM gives first-order transitions for every $q>2$ (Kihara *et al.* 1954, Mittag and Stephen 1974). However, we know that MFT and the Bethe–Peierls approximation (which is equivalent to the treatment of Bethe lattices with translationally invariant probability measures) belong to the same class of approximations (Domb 1960), the latter being an improvement of the former. Therefore, as pointed out by de Magalhães and Tsallis (1981) (see also Wu 1982), the above results seem to be in contradiction. Taking into account our results, the contradiction disappears, since we have shown that $\theta_1=(\sigma-1)/(\sigma+q-1)=T_{BP}$ is a spinodal point, while the actual critical point and the right classification of the transition must be found by looking at the minima of the free energy. Our method is in excellent agreement with MFT, giving first-order transitions for $q>2$, and the same jump of the order parameter at the critical temperature. An improvement can be seen in the value of θ_c, because we have $\Delta\theta_c\equiv\theta_c$ (MFT) $-\theta_c$ (Bethe lattice) >0 (see below for numerical values) with $\Delta\theta_c\to0$ for $\sigma\to\infty$ and/or $q\to\infty$.

Finally, we consider the behaviour of the FPM on regular d-dimensional lattices. Since the effective dimensionality of the Bethe lattices appears to be $d=\infty$ (see e.g. Baxter 1982) we expect that our results are asymptotically exact for high-dimensional standard lattices. In practice we have a good agreement starting from $d=3$. For the three-state FPM on the simple cubic lattice, high- and low-temperature series developments (Straley 1974) suggest singularities respectively at $\theta_1=0.575$ and $\theta_2=0.586$, while Monte Carlo (MC) renormalisation group results (Blöte and Swendsen 1979) and standard MC techniques (Knak Jensen and Mouritsen 1979, Herrmann 1979) give $\theta_c=0.577$. We can simulate this model by setting $q=3$ and $\sigma=5$: MFT gives $\theta_c=0.630$, while our results are $\theta_1=0.571$, $\theta_c=0.587$, $\theta_1=0.589$. The four-state FPM on the $d=4$ hypercubic lattice was studied by means of series expansions by Ditzian and Kadanoff (1979) who found $\theta_1=0.621\pm0.005$, $\theta_c=0.635\pm0.005$, $\theta_2=0.647\pm0.010$. Correspondingly, for $q=4$ and $\sigma=7$, MFT gives $\theta_c=0.662$, while our results are $\theta_1=0.600$, $\theta_c=0.640$, $\theta_2=0.645$. Finally, note that the hysteresis loop limited by θ_1 and θ_2, and spurious transitions from the metastable phases to the stable ones, were observed in MC simulations, too (Blöte and Swendsen 1979, Knak Jensen and Mouritsen 1979).

4. The antiferromagnetic model thermal functions

The procedure introduced in § 2 cannot be generalised to the APM simply by setting $K<0$ in all equations. This can be seen by considering the antiferromagnetic Ising model in its ground state, where the minimal energy and the maximum order realise if every 'up' spin is surrounded by 'down' spins, and *vice versa*. This state can be described by a probability measure μ which, instead of property (i), must satisfy the following one:

(iii) μ is rotationally and two-step translationally invariant on L.

In order to construct μ we proceed as follows. For every T_n let us denote by V_n^e

(V_n^{o}) the set of sites in V_n whose distance $l \geqslant 0$ from the central site u is an even (odd) number. (In the limit $n \to \infty$ this induces a partitioning of V in two sublattices V^{e} and V^{o} such that for every $i \in V^{\text{e}} (i \in V^{\text{o}})$ its adjacent sites belong to V^{o} (V^{e}).) Moreover, let us introduce the following antiferromagnetic ($K < 0$) Potts Hamiltonian with staggered field:

$$-\beta\mathcal{H}_n \equiv K \sum_{\langle ij \rangle \in E_n} \delta_{\nu_i \nu_j} + (H+\tau) \sum_{i \in V_n^{\text{e}}} \delta_{\nu_i 1} + (H-\tau) \sum_{i \in V_n^{\text{o}}} \delta_{\nu_i 1} \tag{21}$$

which generates a probability measure μ_n on T_n. In the limit $n \to \infty$ property (iii) of μ will be obtained by requiring that

$$\lim_{n\to\infty} \mu_n(\nu_u = 1) = \lim_{n\to\infty} \mu_n(\nu_w = 1) \tag{22}$$

where $l(u, w) = 2$; while property (ii) is induced by Hamiltonian (21). We need now eight independent elementary probabilities: p_1^{e}, p_{11}^{e}, p_{21}^{e}, p_{22}^{e}, p_1^{o}, p_{11}^{o}, p_{21}^{o}, p_{22}^{o}, where $p_r^x \equiv \mu(\nu_i = r; i \in V^x)$, $p_{rs}^x \equiv \mu(\nu_j = s | \nu_i = r; i \in V^x)$, x is e or o, and $l(i, j) = 1$. The 'even' probabilities are given by

$$\begin{aligned} &p_1^{\text{e}} = 1/[1 + (q-1)\varphi\bar{\varphi}(\tilde{\varphi})], \qquad p_{11}^{\text{e}} = \mathrm{e}^K/a(\tilde{\varphi}), \\ &p_{21}^{\text{e}} = 1/b(\tilde{\varphi}), \qquad p_{22}^{\text{e}} = \mathrm{e}^K \tilde{\varphi}/b(\tilde{\varphi}), \end{aligned} \tag{23}$$

while the 'odd' probabilities are obtained by interchange of φ and $\tilde{\varphi}$. Here φ is a solution of the equation

$$\varphi = \mathrm{e}^{-(H+\tau)} \bar{\varphi}^{\sigma}(\mathrm{e}^{-(H-\tau)} \bar{\varphi}^{\sigma}(\varphi)) \tag{24}$$

which is implied by (22), and

$$\tilde{\varphi} = \tilde{\varphi}(\varphi) \equiv \mathrm{e}^{-(H-\tau)} \bar{\varphi}^{\sigma}(\varphi). \tag{25}$$

The actual free energy, internal energy and entropy are given by

$$\begin{aligned} &\beta\mathcal{F} = \tfrac{1}{2}(\beta\mathcal{F}_{\text{e}} + \beta\mathcal{F}_{\text{o}}), \qquad \beta\mathcal{U} = \tfrac{1}{2}(\beta\mathcal{U}_{\text{e}} + \beta\mathcal{U}_{\text{o}}), \\ &\mathcal{S} = \tfrac{1}{2}(\mathcal{S}_{\text{e}} + \mathcal{S}_{\text{o}}), \end{aligned} \tag{26}$$

where the 'even' ('odd') component functions have the same functional form of (8), (9), (10), but depend on 'even' ('odd') probabilities and $H + \tau$ $(H - \tau)$. The relevant physical properties of the system will be found, in the most natural way, considering the staggered magnetisation and susceptibility:

$$\mathcal{M}_{\text{s}} \equiv -\partial(\beta\mathcal{F})/\partial\tau = \tfrac{1}{2}(p_1^{\text{e}} - p_1^{\text{o}}), \tag{27a}$$

$$\chi_{\text{s}} \equiv \partial^2(\beta\mathcal{F})/\partial\tau^2 = \tfrac{1}{2}(\chi_-^{\text{e}} + \chi_-^{\text{o}}), \tag{27b}$$

where

$$\begin{aligned} &\chi_\pm^x \equiv p_1^x(1 - p_1^x) \frac{(1 \pm \sigma\Delta^x) \pm \Delta^x(1 \pm \sigma\Delta^y)}{1 - \sigma^2\Delta^x\Delta^y}, \\ &\Delta^x \equiv p_{11}^x - p_{21}^x, \end{aligned} \tag{28}$$

and the pair x, y stands for the apices e, o or o, e. We also consider the 'mixed' susceptibility

$$\chi_{\text{m}} \equiv \partial\mathcal{M}_{\text{s}}/\partial H = \tfrac{1}{2}(\chi_+^{\text{e}} - \chi_+^{\text{o}}) \tag{29}$$

and the standard functions

$$\mathcal{M} \equiv -\partial(\beta\mathcal{F})/\partial H = \tfrac{1}{2}(p_1^{\mathrm{e}} + p_1^{\mathrm{o}}),$$
$$\chi \equiv \partial^2(\beta\mathcal{F})/\partial H^2 = \tfrac{1}{2}(\chi_+^{\mathrm{e}} + \chi_+^{\mathrm{o}}). \tag{30}$$

Finally, the $\tau \to 0$ limit brings us back to our original model, as can be seen by noting that Hamiltonian (21) becomes equal to (1) with $K<0$. However, the system will exhibit new properties, since relation (7) is a particular case of the equation we actually use to compute φ:

$$\varphi = \mathrm{e}^{-H}\bar{\varphi}^{\sigma}(\mathrm{e}^{-H}\bar{\varphi}^{\sigma}(\varphi)) \tag{31}$$

which we obtain, together with the auxiliary relation

$$\tilde{\varphi} = \tilde{\varphi}(\varphi) \equiv \mathrm{e}^{-H}\bar{\varphi}^{\sigma}(\varphi), \tag{32}$$

respectively from (24) and (25).

5. The antiferromagnetic model critical properties

We now look for the solutions of equation (31). It is easy to see that relation (7) for $K<0$ admits one and only one solution φ_{P} for any choice of H and K. This means that (31) always admits the solution φ_{P} with associate value $\tilde{\varphi}(\varphi_{\mathrm{P}}) = \varphi_{\mathrm{P}}$. It can be shown that, for sufficiently low temperatures, and certain ranges of H, equation (31) admits two other solutions $\varphi'_{\mathrm{A}} < \varphi_{\mathrm{P}} < \varphi''_{\mathrm{A}}$ such that

$$\varphi'_{\mathrm{A}} \neq \tilde{\varphi}(\varphi'_{\mathrm{A}}) = \varphi''_{\mathrm{A}}, \qquad \varphi''_{\mathrm{A}} \neq \tilde{\varphi}(\varphi''_{\mathrm{A}}) = \varphi'_{\mathrm{A}}. \tag{33}$$

Let us consider the mathematical and physical implications of the solutions above. Since φ_{P} gives

$$p_r^{\mathrm{e}} = p_r^{\mathrm{o}} \equiv p_r, \qquad r = 1, \ldots, q,$$
$$p_{rs}^{\mathrm{e}} = p_{rs}^{\mathrm{o}} \equiv p_{rs}, \qquad r, s = 1, \ldots, q, \tag{34}$$

it corresponds to a one-step translationally invariant probability measure. Correspondingly equations (26) and (30) reduce respectively to (8), (9), (10), (11); the staggered magnetisation and the mixed susceptibility are identically zero; while (27*b*) becomes

$$\chi_{\mathrm{s}} = p_1(1-p_1)[1-(p_{11}-p_{21})]/[1+\sigma(p_{11}-p_{21})] \tag{35}$$

which is recovered using (14) and the generalised fluctuation relation:

$$\chi_{\mathrm{s}} = \sum_{j\in V^{\mathrm{e}}} \mathcal{G}_{ij}(1,1) + \sum_{r=2}^{q} \sum_{j\in V^{\mathrm{o}}} \mathcal{G}_{ij}(1,r). \tag{36}$$

Finally, since the order parameter of our APM is $m \equiv 2|\mathcal{M}_{\mathrm{s}}|$, we see that $m(\varphi_{\mathrm{P}}) = 0$, i.e. φ_{P} describes a disordered state of the system. Now let us limit our considerations to the set of pairs H, K such that equation (31) admits three distinct solutions. The inequalities in (33) imply that 'even' and 'odd' probabilities corresponding to φ'_{A} (φ''_{A}) are not equal; while the equalities mean that φ'_{A} and φ''_{A} differ only because they interchange the values of 'even' and 'odd' probabilities. In terms of probability measures we say that there are two two-step translationally invariant probability measures: each of them gives the other by means of one-step translations. Since we have

$$m(\varphi'_A) = m(\varphi''_A) > 0, \qquad \mathscr{F}(\varphi'_A) = \mathscr{F}(\varphi''_A) < \mathscr{F}(\varphi_P), \tag{37}$$

we see that the system always prefers the ordered states, represented by φ'_A and φ''_A, to the corresponding disordered state. Therefore φ'_A and φ''_A describe the antiferromagnetic phase of the system, which chooses one of the two possible orderings according to the initial conditions ($\tau \to 0^+$ or $\tau \to 0^-$). There is a set of pairs H, K such that $\varphi'_A = \varphi_P = \varphi''_A$: this is a critical line $\theta_c(h)$ in the h, θ plane (see figure 7) which separates the antiferromagnetic phase found above from the paramagnetic phase (the zone where only φ_P exists). Crossing this line at non-zero temperature, the system always undergoes second-order phase transitions. Indeed, $\mathscr{M}_s$ goes to zero continuously approaching the critical line from within, while χ_s diverges all along $\theta_c(h)$.

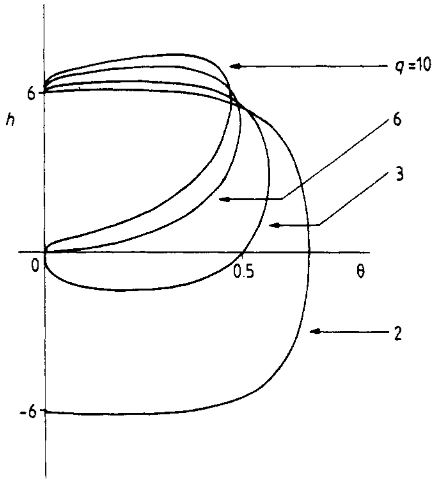

Figure 7. Phase diagrams of the APM in the h, θ plane. The curves are drawn for $\sigma = 5$ and the values of q indicated: they are representative of all Bethe lattices with $\sigma > 1$. The critical lines separate the inner antiferromagnetic ordered phase (staggered magnetisation $\mathscr{M}_s \neq 0$) from the paramagnetic phase ($\mathscr{M}_s = 0$). Along these lines the system undergoes second-order phase transitions, because the staggered magnetisation goes to zero continuously, and the staggered susceptibility diverges. Note the upward shift of the critical lines for increasing q: they intersect the θ axis at non-zero temperature only for $q < \sigma + 1$.

Let us consider now the properties of the antiferromagnetic phase. The lower and upper branches, $h_l(\theta)$ and $h_u(\theta)$, of the critical line start from the h axis and join at $h_b \equiv h_l(\theta_b) = h_u(\theta_b)$, where θ_b is the upper bound to the temperatures such that the antiferromagnetic ordering is possible. The lower branch intersects the θ axis at the temperature $\theta_i \equiv \theta_c(0)$. We have

$$\begin{aligned}
h_l(0) &= \begin{cases} -(\sigma+1), & q = 2 \\ 0, & q > 2, \end{cases} \\
h_u(0) &= \sigma + 1, \\
\theta_i &= \begin{cases} (\sigma - q + 1)/(\sigma + 1), & q \leq \sigma + 1, \\ 0, & q > \sigma + 1, \end{cases} \\
\theta_b &= \tfrac{1}{2}[\![-(q-2) + \{(q-2)^2 + 4(q-1)[(\sigma-1)/(\sigma+1)]^2\}^{1/2}]\!].
\end{aligned} \tag{38}$$

For every fixed temperature $0<\theta<\theta_b$, $|\mathcal{M}_s| = \mathcal{M}_s(\varphi'_A) = -\mathcal{M}_s(\varphi''_A)$ increases from 0 at $h_l(\theta)$ to its maximum value at $h_m(\theta)$, then decreasing to 0 at $h_u(\theta)$. Correspondingly $|\chi_m| \to \infty$ for $h \to h_l(\theta)^+$ and for $h \to h_u(\theta)^-$, while the condition $\chi_m = 0$ defines the line of maximum antiferromagnetic ordering (LMO):

$$h_m(\theta) = \tfrac{1}{2}(\sigma+1) + (2\ln\theta)^{-1}[(\sigma-1)\ln(q-1) - (\sigma+1)\ln(\theta+q-2)] \qquad (39)$$

which ends on the critical line at the point θ_b, h_b. In the antiferromagnetic Ising model this line is the symmetry axis of the critical line, since we have $h_m(\theta) \equiv 0$, $\theta_b = \theta_i$, $h_l(\theta) = -h_u(\theta)$. For $q>2$ the LMO follows the upward shift of the antiferromagnetic phase (see figure 7), since $h_m(\theta)>0$. All along the LMO the ordered states can be seen as redistributions of the spin states present in the corresponding (unstable) disordered state, since we find $\mathcal{M}(\varphi'_A) = \mathcal{M}(\varphi''_A) = \mathcal{M}(\varphi_P)$. For every point in the antiferromagnetic phase not lying on $h_m(\theta)$ this is not true, because $\mathcal{M}(\varphi'_A) = \mathcal{M}(\varphi''_A) > \mathcal{M}(\varphi_P)$. These results imply the cusp in the magnetisation and the jump in the susceptibility which we observe crossing the critical line at every temperature $0<\theta<\theta_b$. The jump in χ is a decreasing function of the temperature which goes to 0 for $\theta \to \theta_b$. Indeed, crossing the critical line at θ_b, h_b, the functions $\mathcal{M}$ and χ are found to be continuous, the singularity being displayed by a cusp in χ.

At zero temperature we find first-order transitions. The paramagnetic phase for $h<h_l(0)$ and $h>h_u(0)$ is characterised respectively by $\mathcal{M}=0$ and by $\mathcal{M}=1$. For every h such that $h_l(0)<h<h_u(0)$ the system disposes in its best ordered state (BOS). In the BOS we find a sublattice completely filled by the spin state 1, while the remaining spin states distribute randomly on the other sublattice. The entropy of the BOS is $\mathcal{S} = (k/2)\ln(q-1)$, which is non-zero for $q>2$†. The state of the system at $h=0$, $\theta=0$ can be obtained by setting $H=0$, $K=-\infty$ in equation (31). We find the BOS for $q=2$; a state characterised by incomplete ordering (i.e. such that $0<m<1$, see figure 8) for $2<q<\sigma+1$; and a paramagnetic state described by $\varphi_P=1$ for $q \geqslant \sigma+1$.

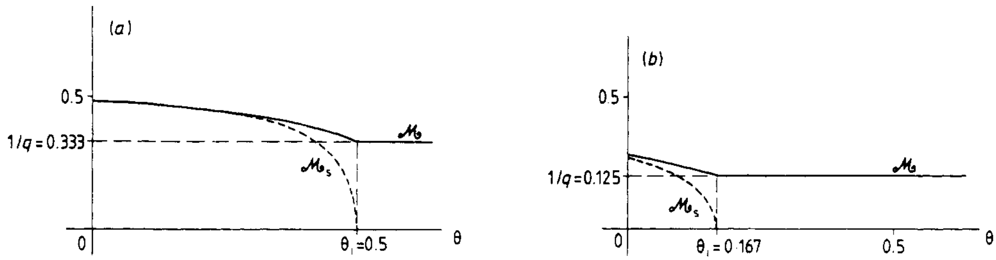

Figure 8. Standard (full line) and staggered (broken line) magnetisations of the APM at $h=0$ for $2<q<\sigma+1$. Note the presence of the spontaneous magnetisation below θ_i, the cusp of $\mathcal{M}$ at θ_i, and the incomplete ordering $\mathcal{M}_s<0.5$ at $\theta=0$. Only the staggered magnetisation corresponding to φ'_A has been plotted: the other ordering gives a line symmetric to this one with respect to the θ axis. The cases $\sigma=5$, $q=3$ and $q=5$, are represented respectively in (*a*) and (*b*).

Remark that, since every pair $H \neq \pm\infty$, $K=-\infty$ is mapped into $h=0$, $\theta=0$, we can find other states in this point, if it is reached with a behaviour of the type (a) $h = \text{constant}/\ln\theta$. For $q=2$ this result is not relevant because only the solution φ_P, i.e. the unstable state, is modified going to $h=0$, $\theta=0$ along (a)-lines (the same

† This is a general property of the APM on bipartite lattices (i.e. formed by two sublattices with the properties of V^e and V^o in the Bethe lattices). See e.g. Berker and Kadanoff (1980).

happens in the FPM for every q). However, for $q>2$, (a)-lines give rise to stable states, all of which are found at the point $h=0$, $\theta=0$. As a matter of fact, note that along $h_l(\theta)$, that is an (a)-line for $q>2$, $q\neq\sigma+1$, and small θ, we reach a state described by $\varphi'_A=\varphi_P=\varphi''_A=(\sigma-1)/(q-2)$.

Finally, let us note that in the APM, for $2<q<\sigma+1$ and $h=0$, the condition $h_m(\theta)>0$ implies $\mathcal{M}(\varphi'_A)=\mathcal{M}(\varphi''_A)>\mathcal{M}(\varphi_P)=1/q$ for every $\theta<\theta_i$: the system exhibits a spontaneous magnetisation (figure 8).

The APM has already been treated on planar lattices (see e.g. Schick and Griffiths 1977, Grest and Banavar 1981, Nightingale and Schick 1982), and on $d>2$ lattices with which we compare the present solution. The antiferromagnetic Ising model on Bethe lattices was solved by Katsura and Takizawa (1974), who found the critical line and the associated cusp in the magnetisation. They also evaluated the zero-field standard and staggered susceptibilities and showed that at θ_i the former has a cusp, while the latter diverges. All these results coincide with ours for $q=2$. The zero-field three-state Potts model has been studied by Oguchi *et al* (1982) with an extension of the effective Hamiltonian method. For antiferromagnetic interactions they find a second-order phase transition at a temperature equal to our θ_i for $q=3$ (the structure of the ordered phase is the same as in the present solution). Banavar *et al* (1980) made MC simulations of the three- and four-state zero-field APM on the simple cubic lattice. In both cases they find a second-order transition to an antiferromagnetic ordered phase, which is characterised by incomplete ordering at zero temperature. The relative critical temperatures seem to be in agreement with our results for $\sigma=5$, i.e. $\theta_i=0.500$ for $q=3$, and $\theta_i=0.333$ for $q=4$. They also find that for $q=5$ the system is always in a paramagnetic state, while we find an ordered phase below $\theta_i=0.167$: in our model the antiferromagnetic phase at $h=0$, $\theta\neq0$ disappears for $q\geqslant6$. The metastable 'plastic crystal' phase, observed by Banavar *et al* (1980) at temperatures below θ_i, cannot be observed in our case because it is removed by the limits $\tau\to0^\pm$. However, it can appear if we relax this condition: let us set $\tau\equiv0$ and cross the critical line by suddenly quenching the system. For $q>2$ a mixing of the two possible orderings can be realised without increase of the internal energy and entropy. The system subdivides in domains (each one described by φ'_A or φ''_A) whose collective properties can give rise to the above-mentioned phase.

6. Conclusions

We have introduced a formalism which has allowed us to solve the Potts model on Bethe lattices. Moreover, by means of it we have a good reproduction (and probably good foresight) of the behaviour of this model on real lattices, as can be seen by noting that the well known main properties of the system are recovered.

One of the main features of the q-state FPM is the existence of a value q_c, depending on the dimensionality d, such that the system undergoes a second-order phase transition at the critical point for $q\leqslant q_c$, and a first-order one for $q>q_c$. It is known that $q_c(2)=4$, $2<q_c(3)<3$, $q_c(4)=2$. In our case $q_c=2$, which agrees with mean field theory of the FPM and with results obtained on standard lattices with $d\geqslant3$.

At zero external field we find two spinodal points on the two sides of the critical point (all of them coincide for $q=2$). We suggest that their possible existence on standard lattices could be responsible for the failures of real space renormalisation groups to find first-order transitions in the FPM (see e.g. Burkhardt *et al* 1976, Dasgupta

1977, den Nijs and Knops 1978, den Nijs 1979a, b). In other words, we think that the renormalisation transformations can be more sensitive to the long-range correlations associated with the spinodal points rather than to the critical point. The first-order phase transitions could be seen if the spinodal points are deleted: this can be obtained by creating local disorder, e.g. introducing vacancies in the system. This interpretation can explain why only recently Nienhuis *et al* (1979, 1980a, b, 1981) and Andelman and Berker (1981) have found first-order transitions applying the Niemijer–van Leeuwen RG and the Migdal–Kadanoff RG to a lattice-gas diluted FPM.

Finally, we used our $h \neq 0$ results to draw the phase diagram of the FPM, which explains the differences between the Ising and Potts models at $h = 0$, and shows their similarity in the larger context $h \neq 0$.

As regards the APM, not much information is known about its behaviour on real lattices. Rescaling arguments support the existence of a cut-off value $q_0(d)$ such that for $q > q_0$ there is no ordered phase at $h = 0$ (Berker and Kadanoff 1980). It is expected (see also Wu 1982) that, for fixed dimensionality d, the critical temperature $\theta_c(h; q)$ at $h = 0$ decreases from $\theta_c(0; 0) = 1$ to $\theta_c(0; q_0) > 0$, and is zero for every $q > q_0$.

On Bethe lattices ($d = \infty$) we find $\theta_c(0; q) = (\sigma - q + 1)/(\sigma + 1)$ and $q_0 = \sigma + 1$, i.e. the jump is missing and q_0 depends on the coordination number. However, note that we find a line of critical points (corresponding to second-order phase transitions for every $q \geqslant 2$) in the h, θ plane, not an isolated singularity at $h = 0$: the ordered phase exists for $q > q_0$, too, but it appears at $h > 0$.

In conclusion, let us remark that our results in the APM can be used as a guideline only for bipartite lattices. Introducing the symmetric APM (with fields $H_r \geqslant 0$ acting on each state $r = 1, \ldots, q$) and/or more complicated staggerings, one should look for n-step translationally invariant probability measures and study their properties.†

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Note added. We have learned recently that I Ono has obtained the same results at $h = 0$ in a different context.

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† The three-state APM on the triangular lattice (which was studied by Schick and Griffiths (1977)) is, e.g., a system described by a three-step translationally invariant probability measure on a tripartite lattice.

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